Calculus 1
Midterm Exam – Solutions
October 1, 2025 (18:30 – 20:30)

university of groningen

---

1) Prove using the (ε, δ)-definition that $\lim_{x\to 3} \frac{x^2-9}{x-3} = 6$.

Solution. Let $\varepsilon > 0$ be arbitrary and take $\delta = \varepsilon$. Then, $0 < |x-3| < \delta$ implies that

$$\left|\frac{x^2-9}{x-3} - 6\right| = \left|\frac{(x-3)(x+3)}{x-3} - 6\right| = |(x+3)-6| = |x-3| < \delta = \varepsilon.$$

Thus, $\lim_{x\to 3} \frac{x^2-9}{x-3} = 6$.

2) Apply l'Hospital's rule to evaluate the following limit: $\lim_{x\to 0^+} \left(\frac{\tan x}{x}\right)^{1/x^2}$. Indicate which rules of differentiation are being used in each step.

Solution. The limit in question has an indeterminate form of type "1^∞" since $\lim_{x\to 0^+}(\tan x)/x = 1$ and $\lim_{x\to 0^+} \frac{1}{x^2} = \infty$. Notice however that moving $(\tan x)/x$ to the exponent by writing

$$\left[\frac{\tan x}{x}\right]^{\frac{1}{x^2}} = \exp\left(\frac{1}{x^2}\ln\frac{\tan x}{x}\right),$$

where $\exp(x) = e^x$ is used to keep things readable, the exponent has an indeterminate form as desired, specifically "$0/0$". We note that we are allowed to do this because when x is near 0, $(\tan x)/x$ is near 1, so the value of $\ln \frac{\tan x}{x}$ exists. Recall that if f is continuous at b and $\lim_{x\to a} g(x) = b$, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(b).$$

[This is Theorem 8 on page 120 of the textbook.] In our case, $f(x) = e^x$ is continuous everywhere, hence, if $\lim_{x\to 0} \frac{\ln \frac{\sin x}{x}}{x^2}$ exists, then we can compute the limit by the above method. The limit of the exponent as $x \to 0^+$ can by found using l'Hospital's Rule thrice. At first, we obtain

$$\ln L = \lim_{x\to 0^+} \frac{\ln\frac{\tan x}{x}}{x^2} \overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{\frac{x}{\tan x}\frac{x\sec^2 x - \tan x}{x^2}}{2x} = \lim_{x\to 0^+} \frac{x\sec^2 x - \tan x}{2x^2\tan x}.$$

Above we used the Chain Rule, the Quotient Rule, the derivatives $(\ln x)' = 1/x$, $(\tan x)' = \sec^2 x$, and the Power Rule $(x^n)' = nx^{n-1}$. At second, we get

$$\ln L = \lim_{x\to 0^+} \frac{x\sec^2 x - \tan x}{2x^2\tan x} \overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{\sec^2 x + 2x\tan x\sec^2 x - \sec^2 x}{4x\tan x + 2x^2\sec^2 x} = \lim_{x\to 0^+} \frac{\tan x \sec^2 x}{2\tan x + x\sec^2 x}.$$

Here we used the Difference Rule, the Product Rule, the Chain Rule, the derivatives $(\tan x)' = \sec^2 x$, $(\sec x)' = \tan x \sec x$, and the Power Rule $(x^n)' = nx^{n-1}$. At third, we arrive at

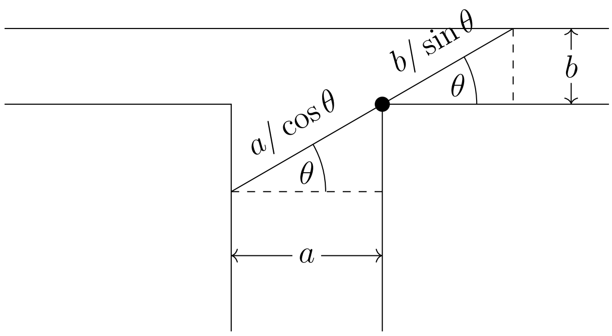
$$\ln L = \lim_{x\to 0^+} \frac{\tan x\sec^2 x}{2\tan x + x\sec^2 x} \overset{\text{l'H}}{=} \lim_{x\to 0^+} \frac{\sec^4 x + 2\tan^2 x\sec^2 x}{3\sec^2 x + 2x\tan x\sec^2 x} = \frac{1}{3},$$

where we used the Constant Multiple Rule, the Sum Rule, the Product Rule, the Chain Rule, and the derivatives $(\tan x)' = \sec^2 x$, $(\sec x)' = \tan x \sec x$, $(x)' = 1$. The last equality follows by direct substitution. Therefore we have

$$L = \lim_{x\to 0^+} \left[\frac{\tan x}{x}\right]^{\frac{1}{x^2}} = \exp\left(\lim_{x\to 0^+} \frac{\ln \frac{\tan x}{x}}{x^2}\right) = e^{1/3} = \sqrt[3]{e}.$$

3) Determine the length of the longest ship that can turn the 90° corner at a T-shaped junction of two completely straight canals of width a and b. You may assume that the width of the ship is negligible (i.e. the ship is a line segment).

Solution. Let us draw a figure of the canals and the ship (tightly) turning the corner:



By using trigonometry we see that for a fixed turning angle $\theta \in (0, \pi/2)$ the longest ship that fits the waterways is of length

$$L(\theta) = \frac{a}{\cos\theta} + \frac{b}{\sin\theta}$$

In order for the ship to turn all the way, we need a ship whose length is *at most* $L(\theta)$ for every angle $0 < \theta < \pi/2$. In other words, we need to find the minimum of $L(\theta)$.

The derivative of $L(\theta)$ reads

$$L'(\theta) = \frac{a\sin\theta}{\cos^2\theta} - \frac{b\cos\theta}{\sin^2\theta}.$$

Solving $L'(\theta) = 0$ for θ, we find that $\tan\theta = \sqrt[3]{b/a}$. Furthermore, the derivative changes sign from negative to positive at this point. To express $\cos\theta$ and $\sin\theta$ in terms of $\tan\theta$, we may use trigonometric identities, e.g. $\cos^2\theta = 1/(1+\tan^2\theta)$ and $\sin^2\theta = 1-\cos^2\theta$. From these, we obtain that if $\tan\theta = \sqrt[3]{b/a}$, then

$$\frac{1}{\cos^2\theta} = 1 + \left(\frac{b}{a}\right)^{2/3}, \quad \frac{1}{\sin^2\theta} = 1 + \left(\frac{a}{b}\right)^{2/3}$$

and therefore the minimum value L attains is

$$\frac{a}{\cos\theta} + \frac{b}{\sin\theta} = a\left[1 + \left(\frac{b}{a}\right)^{2/3}\right]^{1/2} + b\left[1 + \left(\frac{a}{b}\right)^{2/3}\right]^{1/2} = \left[a^{2/3} + b^{2/3}\right]^{3/2}.$$

In conclusion, the longest ship that can turn the corner is of length $(a^{2/3} + b^{2/3})^{3/2}$.

4) Apply the Mean Value Theorem to $f(x) = x\ln(x) - x$ prove the following

Theorem. If $0 < a < b$, then $\left(\frac{a}{b}\right)^b < \frac{e^a}{e^b} < \left(\frac{a}{b}\right)^a$.

Solution. The function $f(x) = x\ln(x) - x$ is an elementary function and therefore it is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Its derivative reads

$$f'(x) = 1 \cdot \ln(x) + x \cdot \frac{1}{x} - 1 = \ln(x).$$

Therefore the Mean Value Theorem can be applied to $f(x)$ over (a, b). The theorem guarantees the existence of a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

which reads

$$\ln(c) = \frac{[b\ln(b) - b] - [a\ln(a) - a]}{b - a}.$$

The numerator on the right-hand side can be expressed using logarithms as follows

$$[b\ln(b) - b] - [a\ln(a) - a] = \ln(b^b e^{-b} a^{-a} e^a).$$

Also $\ln(x)$ is an increasing function, therefore we have $\ln(a) < \ln(c) < \ln(b)$ due to $a < c < b$.

$$(b - a)\ln(a) < \ln(b^b e^{-b} a^{-a} e^a) < (b - a)\ln(b)$$

or, equivalently,

$$\ln(a^{b-a}) < \ln(b^b e^{-b} a^{-a} e^a) < \ln(b^{b-a}).$$

By exponentiating, we obtain the inequalities

$$a^{b-a} < b^b e^{-b} a^{-a} e^a < b^{b-a}.$$

since e^x is an increasing function. Multiplying everything by $a^a/b^b > 0$ yields

$$\frac{a^b}{b^b} < \frac{e^a}{e^b} < \frac{a^a}{b^a}.$$

This concludes the proof.

5) Use Taylor series to find $f^{(2025)}(0)$ if $f(x) = x^{1675}\ln(x + 1)$.

Solution. On the one hand, we know that $f^{(2025)}(0)$ is the $2025!$ times coefficient c_{2025} of the x^{2025} term in the Maclaurin series of f, i.e.

$$f^{(2025)}(0) = c_{2025} 2025!$$

On the other hand, we may multiply the Maclaurin series of $\ln(1 + x)$ by x^{1675} to find the afore-mentioned coefficient c_{2025} directly. We obtain

$$\begin{aligned} f(x) = x^{1685}\ln(1 + x) &= x^{1675}\left[x - \frac{x^2}{2} + \cdots - \frac{x^{350}}{350} + \dots\right] \\ &= x^{1676} - \frac{x^{1677}}{2} + \cdots - \frac{x^{2025}}{350} + \dots \end{aligned}$$

Therefore we see that $c_{2025} = -1/350$ and thus

$$f^{(2025)}(0) = -\frac{2025!}{350}.$$

6) Find the equation for the curve in the xy-plane that passes through the point $(1, -1)$ if the slope of its tangent line at x is always $3x^2 + 2$.

Solution. Having $\frac{dy}{dx} = 3x^2 + 2$ implies that y has the general form

$$y = \int (3x^2 + 2)\, dx = x^3 + 2x + C,$$

where the indefinite integral was found by using the Sum Rule, Power Rule and. The curve $y = x^3 + 2x + C$ passing through the point $(1, -1)$ means that $-1 = (1)^3 + 2(1) + C = 3 + C$ and therefore $C = -4$. Thus the curve is given by the equation

$$y = x^3 + 2x - 4.$$